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A Pincherle Theorem for Matrix Continued Fractions

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Pincherle theorems equate convergence of a continued fraction to existence of a recessive solution of the associated linear system. Matrix continued fractions have recently been used in the study of singular potentials in high energy physics. The matrix continued fractions and discrete Riccati equations previously studied by the author, which were motivated by discrete control theory, had symplectic coefficient matrices. However, the matrix continued fractions employed by Znojil do not have symplectic structure. The previous definition of a recessive solution is modified to allow extension of the Pincherle theorem to include a wider class of continued fractions. © 1996 Academic Press, Inc.

1. INTRODUCTION

Znojil [14, p. 1959] considered "matrix continued fractions"

$$F_k = I/(A_k + C_k F_{k+1} B_{k+1}), \qquad k = \beta, \beta-1, ..., 1. \tag{1}$$

with $F_{\beta+1} = 0$. If the B_k are nonsingular, then this recurrence can be written as

$$F_k = B_{k+1}^{-1}(C_k F_{k+1} + A_k B_{k+1}^{-1})^{-1}. \tag{2}$$

Matrix methods in continued fractions were used by Schwerdtfeger [12]. Noncommutative continued fractions were studied by Pfluger [10] in the context of a ring with identity and by Fair [7, 8] in a complex Banach Algebra.

As a framework for the matrix continued fractions to be defined here, observe that for $n \times n$ matrices A, B, C, D with real or complex entries, we may define a $2n \times 2n$ matrix $\mathbf{A}$ and a matrix Möbius transformation $T_{\mathbf{A}}(\mathscr{Z})$ by

$$\mathbf{A} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \qquad T_{\mathbf{A}}(\mathscr{Z}) = (A\mathscr{Z} + B)(C\mathscr{Z} + D)^{-1}. \tag{3}$$

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Make the formal definition $T_{\mathbf{A}}(\infty)=AC^{-1}$. Now if the block entries of $\mathbf{A}_k$ are allowed to be dependent on k, we could define a nonlinear recurrence by

$$\mathscr{Z}_k=T_{\mathbf{A}_k}(\mathscr{Z}_{k+1})=(A_k\mathscr{Z}_{k+1}+B_k)(C_k\mathscr{Z}_{k+1}+D_k)^{-1}. \tag{4}$$

Of course, the entries of $\mathbf{A}_k$ could be functions of a real variable x or of a complex variable z. Also, the subscripts k need not be restricted to be integers, but can be any real numbers spaced one unit apart; in particular, Example 15 of [5] allows computation of ratios of Bessel functions $J_{\nu+1}(x)/J_\nu(x)$ by means of continued fractions. More generally, the $\mathbf{A}_k$ could be functions of points t_k as well as being functions of a real variable x, or a complex variable z, or other parameters. We will assume throughout that the $\mathbf{A}_k$ are nonsingular.

There are two standard ways of defining the *approximants* of a continued fraction. For m fixed, we could define the kth approximant as the value of $\mathscr{Z}_m$ obtained by following this recurrence back from $\mathscr{Z}_{k+1}=\infty$ or, alternatively, from $\mathscr{Z}_{k+1}=0$. The latter choice was used by Pfluger [10] and Znojil [14]. The continued fraction is said to *converge* if this can be done for large k and the resulting sequence of approximants converges. For the case where all our A_k are 0 and all our C_k are nonsingular, as they are in the studies by Wall [13], Pfluger [10], Fair [7, 8], and Znojil [14], these two methods are equivalent, although the sequences of approximants are shifted by one [13, 9]. Indeed, the approximants starting at $\mathscr{Z}_{k+2}=\infty$ would give $\mathscr{Z}_{k+1}=0$ and the question of convergence of the matrix continued fraction is equivalent to coming back from $\mathscr{Z}=\infty$.

In the next section we make a more general definition of convergence for arbitrary matrix continued fractions with the $\mathbf{A}_k$ nonsingular. That definition facilitates development of a Pincherle theorem.

2. Matrix Continued Fractions

Assume throughout that the $2n\times 2n$ matrices $\mathbf{A}_k$ of (3) are nonsingular. The $n\times n$ blocks A_k, B_k, C_k, D_k can have real or complex entries. We now make our definition of the sequence of approximants. For $k\geqslant m$, introduce the notation

$$\begin{bmatrix} P_k & Q_k \\ R_k & S_k \end{bmatrix}=\mathbf{A}_m\mathbf{A}_{m+1}\cdots\mathbf{A}_k \tag{5}$$

and formally define the sequence of *approximants* as

$$T_{\mathbf{A}_m\mathbf{A}_{m+1}\cdots\mathbf{A}_k}(\infty)=P_kR_k^{-1}, \qquad \text{for} \quad k=m, m+1, ... \tag{6}$$

The continued fraction

$$\{T_{\mathbf{A}_m \mathbf{A}_{m+1} \cdots \mathbf{A}_k}(\infty)\} \tag{7}$$

is said to *converge* if the *partial denominators* R_k are nonsingular for large k and the sequence $\{P_k R_k^{-1}\}$ has a limit. Note that the formal composite

$$T_{\mathbf{A}_m \mathbf{A}_{m+1} \cdots \mathbf{A}_k}(\mathscr{Z}) \tag{8}$$

is an extension of the functional composite

$$T_{\mathbf{A}_m} \circ T_{\mathbf{A}_{m+1}} \circ \cdots \circ T_{\mathbf{A}_k}(\mathscr{Z}), \qquad k = m, m+1, ..., \tag{9}$$

in the sense that when $\mathscr{Z}$ is in the domain of the functional composite, then $\mathscr{Z}$ is in the domain of the formal composite and the functions agree [4]. Thus convergence of classical functional composite continued fractions implies convergence of the formal composite continued fractions.

Since $\mathbf{A}_m \mathbf{A}_{m+1} \cdots \mathbf{A}_k = (\mathbf{A}_m \mathbf{A}_{m+1} \cdots \mathbf{A}_{k-1})\, \mathbf{A}_k$ we have

$$\begin{bmatrix} P_k & Q_k \\ R_k & S_k \end{bmatrix} = \begin{bmatrix} P_{k-1} & Q_{k-1} \\ R_{k-1} & S_{k-1} \end{bmatrix} \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} \tag{10}$$

which also holds for $k = m$ if we make the definitions

$$P_{m-1} = I, \quad Q_{m-1} = 0, \quad R_{m-1} = 0, \quad S_{m-1} = I. \tag{11}$$

The *key idea* which connects the theory of continued fractions to the theory of linear recurrences is obtained by a simple idea, namely, the transpose of (10), is

$$\begin{bmatrix} P_k^T & R_k^T \\ Q_k^T & S_k^T \end{bmatrix} = \mathbf{A}_k^T \begin{bmatrix} P_{k-1}^T & R_{k-1}^T \\ Q_{k-1}^T & S_{k-1}^T \end{bmatrix}. \tag{12}$$

Thus the pair $Y_1(k) = P_k^T$, $Z_1(k) = Q_k^T$ and the pair $Y_2(k) = R_k^T$, $Z_2(k) = S_k^T$ are solutions of the system

$$\begin{bmatrix} Y(k) \\ Z(k) \end{bmatrix} = \mathbf{M}(k) \begin{bmatrix} Y(k-1) \\ Z(k-1) \end{bmatrix} \tag{13}$$

where

$$\mathbf{M}(k) \equiv \mathbf{A}_k^T = \begin{bmatrix} A_k^T & C_k^T \\ B_k^T & D_k^T \end{bmatrix} \equiv \begin{bmatrix} E_k & F_k \\ G_k & H_k \end{bmatrix}. \tag{14}$$

Relabel $m-1$ as l. Then the initial conditions (11) on the P_k, Q_k, etc., become

$$Y_1(l)=I, \qquad Z_1(l)=0, \qquad Y_2(l)=0, \qquad Z_2(l)=I. \tag{15}$$

Set

$$\mathbf{X}(k)=\begin{bmatrix} Y(k) \\ Z(k) \end{bmatrix}, \qquad \text{then} \quad \mathbf{X}(k)=\mathbf{M}(k)\,\mathbf{X}(k-1). \tag{16}$$

The assumption that $\mathbf{M}(k)$ is nonsingular and the initial conditions on $\mathbf{X}_1$ and $\mathbf{X}_2$ make $[\mathbf{X}_1(l)\ \mathbf{X}_2(l)]=I_{2n}$ and the pair $\mathbf{X}_1$, $\mathbf{X}_2$ is a *fundamental solution*, i.e., a *basis*, in the sense that any $2n\times n$ solution $\mathbf{X}$ may be expressed uniquely as $\mathbf{X}=\mathbf{X}C_1+\mathbf{X}C_2$ for $n\times n$ constant matrices C_1 and C_2. Note that the solution space of $2n\times n$ solutions $\mathbf{X}(k)$ together with right multiplication by $n\times n$ constant matrices is a right unitary module and is not a vector space because, except for the case of $n=1$, the scalars come from a noncommutative ring with identity instead of a field. The assumption of nonsingularity of the $\mathbf{M}(k)$ makes this module have well defined dimension of 2.

We say that $2n\times n$ solutions $\mathbf{X}_0(k)$ and $\mathbf{X}(k)$ of (16) are *linearly independent* if the only $n\times n$ constant matrices Γ_0 and Γ such that

$$\mathbf{X}_0(k)\,\Gamma_0+\mathbf{X}(k)\,\Gamma\equiv 0 \tag{17}$$

are $\Gamma_0=0$ and $\Gamma=0$. This generalizes the definition used in [6, p. 6].

This conversion of the study of approximants of a continued fraction to the study of linear recurrences makes possible the investigation of the nonlinear recurrences of functional composites of Möbius transformations, i.e., discrete matrix Riccati equations [1], by linear methods.

Classical theory, as well as Pfluger's matrix continued fractions, employed linear three term recurrence relations, but as in differential equations it is more natural and convenient to deal with "first order linear systems." For completeness and for the application of Znojil [14], continued fractions with our blocks $A_k=0$ and nonsingular C_k are related to three term recurrences as in the scalar case and the special matrix case of Pfluger [10].

PROPOSITION 1. *Suppose $A_k\equiv 0$ and the C_k are nonsingular. Then the partial numerators P_k and partial denominators R_k of the continued fraction* (7) *must satisfy the three term recurrence*

$$U_{k+1}C_{k+1}^{-1}=U_{k-1}B_k+U_kC_k^{-1}D_k \tag{18}$$

with the initial conditions

$$P_{m-1} = I, \quad P_m = 0, \quad R_{m-1} = 0, \quad R_m = C_m. \tag{19}$$

Proof. Because of the assumption that $E_k = 0$ we have $Y(k) = F_k Z(k-1)$. Solve this for Z and replace Z in $Z(k) = G_k Y(k-1) + H_k Z(k-1)$ to obtain the three term recurrence

$$F_{k+1}^{-1} Y(k+1) = G_k Y(k-1) + H_k F_k^{-1} Y(k). \tag{20}$$

Take the transpose of both sides of this recurrence and set $U_k = Y^T(k)$ for (18). ∎

The assumption of convergence of the matrix continued fraction (7) is equivalent to the assumptions that $Y_2(k)$ is nonsingular for large k and there exists a matrix limit Γ_m such that

$$P_k R_k^{-1} = Y_1^T(k)(Y_2^T(k))^{-1} \to \Gamma_m. \tag{21}$$

If we relabel Γ_m^T as $\Omega(l)$ with $l = m-1$, then transposing the above gives

$$Y_2^{-1}(k)\, Y_1(k) \to \Omega(l). \tag{22}$$

We summarize these observations as follows:

PROPOSITION 2. *Assume that the $2n \times 2n$ matrix coefficients $\mathbf{A}_k$ are nonsingular, $\mathbf{M}_k = \mathbf{A}_k^T$, $l = m-1$, and $\mathbf{X}_1(k)$, $\mathbf{X}_2(k)$ are the solutions of* (16) *defined by the initial conditions* (15). *The matrix continued fraction* (7) *converges as $k \to \infty$ to a matrix limit Γ_m if and only if $Y_2(k)$ is nonsingular for large k and $Y_2^{-1}(k)\, Y_1(k) \to \Omega(l) = \Gamma_m^T$.*

Note that $\mathbf{X}$ is a $2n \times n$ solution of (16) if and only if $\mathbf{X} = \mathbf{X}_1 C_1 + \mathbf{X}_2 C_2$ for some $n \times n$ constant matrices C_1 and C_2. Hence convergence of the continued fraction (7) implies the limit (22) and

$$Y_2^{-1}(k)\, Y(k) = Y_2^{-1}(k)\, Y_1(k)\, C_1 + C_2 \to \Omega(l)\, C_1 + C_2. \tag{23}$$

Thus the solution $\mathbf{X}_0$ chosen as

$$\mathbf{X}_0 = \mathbf{X}_1 - \mathbf{X}_2 \Omega(l) \tag{24}$$

would have the properties $Y_0(k) = Y_1(k) - Y_2(k)\, \Omega(l)$, $Y_0(l) = I - 0 = I$ is nonsingular, $Y_2(k)$ is nonsingular for large k, and

$$Y_2^{-1}(k)\, Y_0(k) = Y_2^{-1}(k)\, Y_1(k) - \Omega(l) \to \Omega(l) - \Omega(l) = 0. \tag{25}$$

Now in what sense is $\mathbf{X}_0$ recessive? Suppose that $\mathbf{X}$ is a $2n \times n$ solution such that the $2n \times 2n$ partitioned matrix $[\mathbf{X}_0(k)\ \ \mathbf{X}(k)]$ is nonsingular; i.e., $\mathbf{X}$ and $\mathbf{X}_0$ are *linearly independent* in the solution module. Note that nonsingularity at one value of k implies nonsingularity at all k because the $\mathbf{M}(k)$ are all nonsingular. Since $\mathbf{X}_1$, $\mathbf{X}_2$ is a basis, there exist $n \times n$ constant matrices C_1 and C_2 such that $\mathbf{X} \equiv \mathbf{X}_1 C_1 + \mathbf{X}_2 C_2$. Hence $Y(k) = Y_1(k)\, C_1 + Y_2(k)\, C_2$ and $Y_0(k) = Y_1(k) - Y_2(k)\, \Omega(l)$ with $Y_2^{-1}(k)\, Y_0(k) \to 0$. Thus $Y_2^{-1}(k)\, Y(k) = Y_2^{-1}(k)\, Y_1(k)\, C_1 + C_2 \to \Omega(l)\, C_1 + C_2$. Now if we knew that $\Omega(l)\, C_1 + C_2$ were nonsingular, then we could conclude nonsingularity of $Y(k)$ for large k. Suppose that u is an n vector such that $(\Omega(l)\, C_1 + C_2)\, u = 0$. Nonsingularity of $[\mathbf{X}_0(k)\ \ \mathbf{X}(k)]$ implies nonsingularity of

$$\begin{bmatrix} Y_0(l) & Y(l) \\ Z_0(l) & Z(l) \end{bmatrix} = \begin{bmatrix} I & C_1 \\ -\Omega(l) & C_2 \end{bmatrix}. \tag{26}$$

However,

$$\begin{bmatrix} I & C_1 \\ -\Omega(l) & C_2 \end{bmatrix} \begin{bmatrix} -C_1 u \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{27}$$

since $\Omega(l)\, C_1 u + C_2 u = 0$. Thus $u = 0$, $\Omega(l)\, C_1 + C_2$ is nonsingular, and $Y(k)$ is nonsingular for large k. Now as in [5] we wish to show that

$$Y^{-1}(k)\, Y_0(k) \to 0. \tag{28}$$

Observe that

$$\begin{aligned} Y^{-1}(k)\, Y_0(k) &= [\, Y_1(k)\, C_1 + Y_2(k)\, C_2]^{-1}\, [\, Y_1(k) - Y_2(k)\, \Omega(l)] \\ &= (\, Y_2^{-1}(k)\, Y_1(k)\, C_1 + C_2)^{-1}\, [\, Y_2^{-1}(k)\, Y_1(k) - \Omega(l)] \\ &\to (\Omega(l)\, C_1 + C_2)^{-1}\, (\Omega(l) - \Omega(l)) = 0 \end{aligned} \tag{29}$$

as we wished to show. We now base our definition of recessive at ∞ upon the above properties of $\mathbf{X}_0$.

We say that a $2n \times n$ solution $\mathbf{X}_0$ is *recessive at* ∞ if

1. $\mathbf{X}_0$ has full column rank of n.

2. If $\mathbf{X} = [\begin{smallmatrix} Y \\ Z \end{smallmatrix}]$ is any $2n \times n$ solution such that $[\mathbf{X}_0\ \ \mathbf{X}]$ is nonsingular, i.e., $\mathbf{X}_0$ and $\mathbf{X}$ are linearly independent, then $Y(k)$ is nonsingular for large k and

$$Y^{-1}(k)\, Y_0(k) \to 0. \tag{30}$$

With this definition of a recessive solution at ∞ we are now ready to state a Pincherle theorem for these matrix continued fractions.

THEOREM 3. *Assume nonsingular* $\mathbf{A}_k$. *A necessary and sufficient condition for convergence of the continued fraction*

$$\{T_{\mathbf{A}_m \mathbf{A}_{m+1} \cdots \mathbf{A}_k}(\infty)\}, \qquad k = m, m+1, ... \tag{31}$$

is that there exists a recessive solution at ∞,

$$\mathbf{X}_0 = \begin{bmatrix} Y_0 \\ Z_0 \end{bmatrix} \tag{32}$$

of (16) *with* $Y_0(m-1)$ *nonsingular. Furthermore, if the continued fraction converges, then it converges to*

$$\Gamma_m = -[Z_0(m-1)\, Y_0^{-1}(m-1)]^T. \tag{33}$$

Proof. We have proven necessity. In order to show sufficiency, assume that there exists a recessive solution with $Y_0(l)$ nonsingular. Then there exist C_1 and C_2 such that $\mathbf{X}_0 = \mathbf{X}_1 C_1 + \mathbf{X}_2 C_2$. Thus $C_1 = Y_0(l)$ is nonsingular, $C_2 = Z_0(l)$, and $Y_0(k) = Y_1(k)\, C_1 + Y_2(k)\, C_2$. Now

$$\begin{bmatrix} Y_0(l) & Y_2(l) \\ Z_0(l) & Z_2(l) \end{bmatrix} = \begin{bmatrix} C_1 & 0 \\ C_2 & I \end{bmatrix} \tag{34}$$

with C_1 nonsingular implies a full rank $[\mathbf{X}_0(l)\ \mathbf{X}_2(l)]$. However, $\mathbf{X}_0$ recessive allows us to conclude that $Y_2(k)$ is nonsingular for large k and $Y_2^{-1}(k)\, Y_0(k) \to 0$, as $k \to \infty$. Thus

$$Y_2^{-1}(k)\, Y_0(k) = Y_2^{-1}(k)\, Y_1(k)\, C_1 + C_2 \to 0 \tag{35}$$

implies that $Y_2^{-1}(k)\, Y_1(k) \to -C_2 C_1^{-1}$ and the continued fraction converges to $\Gamma_m = -(C_2 C_1^{-1})^T = -[Z_0(l)\, Y_0^{-1}(l)]^T$. Thus the Pincherle theorem is established for this family of matrix continued fractions. ▮

We now show that recessive solutions with $Y_0(l)$ nonsingular are essentially unique.

THEOREM 4. *Assume nonsingular* $\mathbf{A}_k$.

(i) *If* $\mathbf{X}_0$ *is recessive and* K *is any nonsingular* $n \times n$ *constant matrix, then* $\mathbf{X}_0 K$ *is also recessive.*

(ii) *If* $\mathbf{X}_0$ *and* $\hat{\mathbf{X}}_0$ *are recessive solutions with* $Y_0(l)$ *and* $\hat{Y}_0(l)$ *nonsingular, then there exists a nonsingular constant matrix* K *such that*

$$\hat{\mathbf{X}}_0 \equiv \mathbf{X}_0 K. \tag{36}$$

Proof. For part (i), let $\mathbf{X}_3 = \mathbf{X}_0 K$ and let $\mathbf{X}$ be a solution with $[\mathbf{X}_3(l)\ \ \mathbf{X}(l)]$ nonsingular. Then

$$[\mathbf{X}_0(l) \quad \mathbf{X}(l)] \begin{bmatrix} K & 0 \\ 0 & I \end{bmatrix}$$

is nonsingular and $[\mathbf{X}_0(l)\ \ \mathbf{X}(l)]$ is nonsingular. Because $\mathbf{X}_0$ is recessive we have $Y(k)$ nonsingular for large k, $Y^{-1}(k)\, Y_0(k) \to 0$, and $Y^{-1}(k)\, Y_3(k) \to 0$ as $k \to \infty$. Thus X_3 is recessive.

For part (ii), the solutions $\mathbf{X}_0(k)\, Y_0^{-1}(l)$ and $\hat{\mathbf{X}}_0(k)\, \hat{Y}_0^{-1}(l)$ are recessive solutions with first components of I at l. If we can show that these solutions have the same second component at l, then

$$\mathbf{X}_0(k)\, Y_0^{-1}(l) \equiv \hat{\mathbf{X}}_0(k)\, \hat{Y}_0^{-1}(l)$$

and $\hat{\mathbf{X}}_0(k) \equiv \mathbf{X}_0(k)\, K$ for $K = Y_0^{-1}(l)\, \hat{Y}_0(l)$. Thus it suffices to show that if $\mathbf{X}_0$ and $\hat{\mathbf{X}}_0$ are recessive solutions with $Y_0(l) = I = \hat{Y}_0(l)$, then $Z_0(l) = \hat{Z}_0(l)$. Let $\mathbf{X}_1$ and $\mathbf{X}_2$ satisfy the initial conditions of (15). Then

$$[\mathbf{X}_0(l) \quad \mathbf{X}_2(l)] = \begin{bmatrix} I & 0 \\ Z_0(l) & I \end{bmatrix} \quad \text{and} \quad [\hat{\mathbf{X}}_0(l) \quad \mathbf{X}_2(l)] = \begin{bmatrix} I & 0 \\ \hat{Z}_0(l) & I \end{bmatrix}$$

are nonsingular. Thus $Y_2(k)$ must be nonsingular for large k, $Y_2^{-1} Y_0 \to 0$, and $Y_2^{-1} \hat{Y}_0 \to 0$. Use initial conditions at l to obtain $Y_0(k) = Y_1(k) + Y_2(k)\, Z_0(l)$ and $\hat{Y}_0(k) = Y_1(k) + Y_2(k)\, \hat{Z}_0(l)$. Therefore,

$$Y_2^{-1}(k)\, Y_0(k) = Y_2^{-1}(k)\, Y_1(k) + Z_0(l) \to 0$$

and

$$Y_2^{-1}(k)\, \hat{Y}_0(k) = Y_2^{-1}(k)\, Y_1(k) + \hat{Z}_0(l) \to 0.$$

Thus $Y_2^{-1}(k)\, Y_1(k)$ has a limit. Uniqueness of limits implies that $Z_0(l) = \hat{Z}_0(l)$ as desired. ▮

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